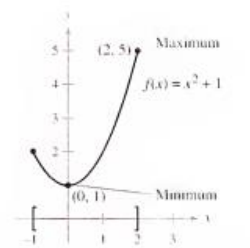
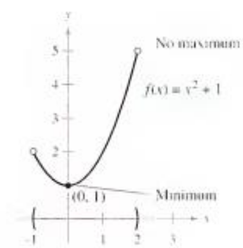
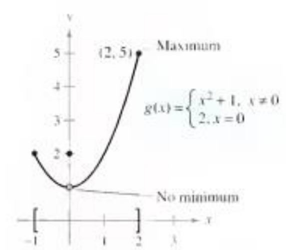
5.1 Extreme Values of Functions

Looking Back: Given the function $f(x) = \begin{cases} \frac{(x^2-4)}{x-2}; & x \neq 2 \\ 1; & x = 2 \end{cases}$, which of the following statements about f is true?

I. F has a limit at x = 2.
II. F is continuous at x = 2
III. F is differentiable at x = 2.

a. I only   b. II only   c. III only   d. I and II   e. I, II, and III

Definition of Extrema	Examples:
Let f be defined on an interval I containing c. 1. $f(c)$ is the **minimum of f on I** when $f(c) \leq f(x)$ for all x in I. 2. $f(c)$ is the **maximum of f on I** when $f(c) \geq f(x)$ for all x in I. The minimum and maximum of a function on an interval are called the **extreme values**, or **extrema**. They call also be called the **absolute minimum or maximum or global minimum or maximum.** Extrema can occur at interior points or endpoints of an interval. **Endpoint extrema** are extrema on the endpoints.	 (a) f is continuous, [−1, 2] is closed.  (b) f is continuous, (−1, 2) is open. 

Extreme Value Theorem: If f is continuous on a closed interval [a, b], then f has both a minimum and a maximum on the interval.

Definition of Relative Extrema:

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f, or you can say that f has a **relative maximum at** $(c, f(c))$.

2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f, or you can say that f has a **relative minimum at** $(c, f(c))$.

The plural of relative maximum is relative maxima, the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called **local maximum and local minimum**, respectively.

Critical Number: Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.

Relative Extrema Occur Only at Critical Numbers: If f has a relative minimum or relative maximum at x = c, then ci s a critical number of f.

Examples:

A. Find all local and global extrema of $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 - 14x + 7$.

1. Find the derivative	a. $f'(x) = x^2 - 5x - 14$
2. Set the derivative equal to zero and solve for x.	b. $0 = x^2 - 5x - 14$ $0 = (x-7)(x+2)$ $x = 7, -2$

B. Find all absolute and relative extrema of $g(x) = \begin{cases} 5 - 2x^2; & x \le 1 \\ x + 2; & x > 1 \end{cases}$

1. Find the derivative	a. $f'(x) = \begin{cases} \frac{d}{dx}(5 - 2x^2, & x < 1 \\ \frac{d}{dx}(x+2), & x > 1 \end{cases}$ b. $f'(x) = \begin{cases} -4x, & x \le 1 \\ 1, & x > 1 \end{cases}$
2. Set the derivative equal to zero and solve for x.	c. The only part that can be set equal to zero is -4x. At -4x = 0, x = 0.
3. What happens at x = 1? Find the right- and left- hand derivatives.	d. $\lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)+2-3}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$ e. $\lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^-} \frac{5-2(1+h)^2-3}{h} =$ $\lim_{h \to 0^+} \frac{-2h(2+h)}{h} = -4$
4. Because the one-sided limits are different, f has no derivative at x = 1, thus, 1 is another critical point of f.	
5. The domain of the function is $(-\infty, \infty)$; therefore there are no endpoints and the only critical points are f(0) = 5 and f(1) = 3.	

C. Finding Extrema on a Closed Interval: Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.

1. Find the derivative.	a. $f'(x) = 12x^3 - 12x^2$
2. Set equal to zero and solve for x.	b. $0 = 12x^2(x-1)$ $x = 0, 1$
3. Then evaluate f at these two critical numbers and the two endpoints.	(see table below)

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = 7$	$f(0) = 0$	$f(1) = -1$ Minimum	$f(2) = 16$ Maximum