4.4 Derivatives of Inverse Trig Functions

Continuity and Differentiability of Inverse Functions: Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.

2. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

The Derivative of an Inverse Function: Let f be a function that is differentiable on an interval I. If f has an inverse function, then g is differentiable at any x for which $f'\big(g(x)\big) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, f'\big(g(x)\big) \neq 0.$$

Example: Let $f(x) = \frac{1}{4}x^3 + x - 1$

a. What is the value of $f^{-1}(x)$, when x = 3?
b. What is the value of $(f^{-1})'(x)$, when x = 3?

**Notice that f is one-to-one and therefore has an inverse function.

a. Because f(2) = 3, we know that $f^{-1}(3) = 2$.
b. Because the function is differentiable and has an inverse function, you can write:

$$f^{-1}(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)}$$

Using $f'(x) = \frac{3}{4}x^2 + 1$

$$f^{-1}(3) = \frac{1}{f'(2)} = \frac{1}{\frac{3}{4}(2^2) + 1} = \frac{1}{4}$$

We can also write the inverse derivative as $\frac{dy}{dx} = \frac{1}{dx/dy}$

Derivatives of Inverse Trig Functions: Let u be a differentiable function of x.

$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$	$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$	$\frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$
$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$	$\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{\lvert u\rvert\sqrt{u^2-1}}$

Examples:

a. $\frac{d}{dx}[\arcsin 2x] = \frac{2}{\sqrt{1-(2x)^2}} = \frac{2}{\sqrt{1-4x^2}}$

b. $\frac{d}{dx}[\arctan 3x] = \frac{3}{1+(3x)^2} = \frac{3}{1+9x^2}$

Find the derivative of $y = \arcsin x + x\sqrt{1-x^2}$

$$\frac{d}{dx}\left(y = \arcsin x + x\sqrt{1-x^2}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{2}\right)(-2x)(1-x^2)^{-\frac{1}{2}} + \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \sqrt{1-x^2} + \sqrt{1-x^2}$$

$$\frac{dy}{dx} = 2\sqrt{1-x^2}$$