4.3 Implicit Differentiation

Most function have been expressed in explicit form. For example: $y = 3x^2 - 5$, the variable y is explicitly written as a function of x.

Implicit form: when the function is not defined explicitly as a function of y in regards to x. For example the explicit equation $y = \frac{1}{x}$ can be written implicitly as $xy = 1$

Implicit Form	Explicit Form	Derivative
$xy = 1$	$y = \frac{1}{x}$	$\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$

When you are unable to solve for y in terms of x, you can still find the derivative dy/dx.

Key to remember: The differentiation is taking place with respect to x. This means that when you differentiation terms involving x along, you can differentiation like normal. When you differentiation terms involving y, you must apply the Chain Rule.

Examples

$\frac{d}{dx}(x^3) = 3x^2$ *The variables agree on the left.	Variables agree; therefore, use Power Rule like usual.
$\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$ *The variables disagree on the left	Variables disagree; therefore, use the Chain Rule.
$\frac{d}{dx}(x + 3y) = 1 + 3\frac{dy}{dx}$	Chain Rule: $\frac{d}{dx}(3y) = 3y'$
$\frac{d}{dx}(xy^2) = \frac{d}{dx}x * y^2 + x * \frac{d}{dx}y^2$ $= (1)y^2 + x\,(2y\,\frac{dy}{dx})$ $= y^2 + 2xy\frac{dy}{dx}$	a. Product Rule b. Chain Rule c. Simplify

Example: Find dy/dx given that $y^3 + y^2 - 5y - x^2 = -4$

1. Differentiation both sides of the equation with respect to x.

$$\frac{d}{dx}(y^3 + y^2 - 5y - x^2 = -4)$$

2. Collect all terms involving dy/dx on one side of the equation and move all other terms to the other side of the equation.

$$3y^2\frac{dy}{dx} + 2y\,\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

$$3y^2\frac{dy}{dx} + 2y\,\frac{dy}{dx} - 5\frac{dy}{dx} = 2x$$

3. Factor out dy/dx.

$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

4. Solve for dy/dx.

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$

Find the Derivatives of a Higher Order

Given $x^2 + y^2 = 25, find\ \frac{d^2y}{dx^2}$

Differentiating each term with respect to x produces:

$$\frac{d}{dx}(x^2 + y^2 = 25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Differentiating a second time with respect to x yields:

$$\frac{d^2y}{dx^2} = \frac{-y(1) - (x)\left(\frac{dy}{dx}\right)}{y^2}$$

Substitute for dy/dx:

$$\frac{d^2y}{dx^2} = \frac{-y(1) - (x)\left(-\frac{x}{y}\right)}{y^2}$$

Simplify: $= -\frac{y^2+x^2}{y^3}$

Simplify more with substitution: $\frac{-25}{y^3}$