3.3 Rules for Differentiation

1. Derivative of a Constant Function: If f is the function with the constant value c, then $\frac{df}{dx} = \frac{d}{dx}(c) = 0$

Proof: If f(x) = c is a function with a constant value c, then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

Example 1: Find the derivative of $f(x) = 34$

2. Power Rule for Integer Powers of x: If n is a positive integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$

Proof: If $f(x) = x^n$, then

$$\frac{d}{dx}(x^n) = \lim_{h \to 0} \frac{(x+h)^n - x^n h}{h}$$
$$= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + h^2 - x^n}{h}$$
$$= \lim_{h \to 0} \frac{nx^{n-1}h + h^2}{h}$$
$$= \lim_{h \to 0} nx^{n-1} + h$$
$$= nx^{n-1}$$

Example 2: Find the derivative of $f(x) = x^3$

3. The Constant Multiple Rule: If u is differentiable function of x and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

Proof:

$$\frac{d}{dx}(cu) = \lim_{h \to 0} \frac{cu(x+h) - cu(x)}{h}$$
$$= c \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$$
$$= c\frac{du}{dx}$$

Example: Find the derivative of $f(x) = 3x^2$

4. The Sum and Difference Rule: If u and v are differentiable functions of x, then their sum and difference are differentiable at every point where u and v are differentiable. At such points, $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$.

Proof: $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

Use the difference quotient for $f(x) = u(x) + v(x)$

$$\frac{d}{dx}[u(x)+v(x)] = \lim_{h\to 0}\frac{[u(x+h)+v(x+h)]-[u(x)+v(x)]}{h}$$

$$= \lim_{h\to 0}\frac{[u(x+h)-u(x)+v(x+h)-v(x)]}{h}$$

$$= \frac{du}{dx}+\frac{dv}{dx}$$

Example: Find f'(x) if $f(x) = x^3 + 6x^2 - \frac{5}{3}x$

5. The Product Rule: The product of two differentiable functions v and v is differentiable, and $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Proof: $\frac{d}{dx}(uv) = \lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x)v(x)}{h}$

To change the fraction into an equivalent one that contains difference quotients for the derivatives of u and v, we subtract and add u(x +h)v(x) in the numerator.

$$= \lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x+h)v(x)+u(x+h)v(x)-u(x)v(x)}{h}$$

Then factor and separate.

$$= \lim_{h\to 0}[u(x+h)\frac{v(x+h)-v(x)}{h}+v(x)\frac{u(x+h)-u(x)}{h}$$

$$= \lim_{h\to 0}u(x+h) * \lim_{h\to 0}\frac{v(x+h)-v(x)}{h}+v(x)\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

$$\frac{d}{dx}(u\pm v) = \frac{du}{dx}\pm\frac{dv}{dx}$$

Example: Find f'(x) if $f(x) = (x^2+1)(x^3+3)$

6. The Quotient Rule: At a point where $v \neq 0$, the quotient $y = {}^{u}/_{v}$ of two differentiable functions is differentiable, and $\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

Horizontal Tangent: To find the horizontal tangent, first find the derivative of the function. Then set the derivative equal to zero and solve for x.

Example: $y = 4x^3 - 6x^2 - 1, y' = 12x^2 - 12x, x = 0, x = 1$