3.1 Derivative of a Function

Derivative: The derivative of the function *f* with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad or \quad \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The derivative a function of x is also a function of x. This "new" function gives the slop of the tangent line to the graph of f at the point (x, f(x)).

Derivatives are used to find instantaneous rate of change of one variable with respect to another.

Differentiation: the process of finding the derivative.

Differentiable: a function is differentiable at x when its derivative exists at x and is differentiable on an open interval (a, b). ←←←SUPER IMPORTANT.

Example 1: Differentiate $f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \lim_{h \to 0} \frac{(3x^3 + 3xh + h^2)h}{h}$$

$$\lim_{h \to 0} (3x^3 + 3xh + h^2) = 3x^2$$

Notation: $f'(x),\ \frac{dy}{dx},\ y',\ \frac{d}{dx}[f(x)],\ D_x[y]$

Alternate Form of Derivative: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ provided this limit exists.

Example 2: Differentiate $f(x) = \sqrt{x}$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$$

$$= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} * \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}$$

$$= \lim_{x \to a} \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})}$$

$$\lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}}$$
$$= \frac{1}{2\sqrt{a}}$$

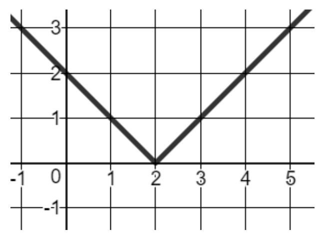
The alternative limit form of the derivative is useful for investigating the relationship between differentiability and continuity.

The existence of the limit in this alternative form requires that one-sided limits exist and are equal. $\lim_{x \to a^-} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$. These one-sided limits are called derivatives from the left and from the right.

Example 3:
The function $f(x) = |x - 2|$ is continuous at x = 2. The one-sided limits, however, are not equal, as shown.

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{|x-2| - 0}{x - 2} = -1$$

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{|x-2| - 0}{x - 2} = 1$$



The function is not differentiable at x = 2 and the graph of the function does not have a tangent line at the point (2, 0).

Example 4: Find the equation of the tangent line to the graph of f at the indicated point.

$$f(x) = x^3 + 2x; \quad (1, 3)$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 + 2(x+h) - (x^3 + 2x)}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - x^3 - 2x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 + 2h}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 + 2 = 3x^2 + 2$$

- Using $f'(x) = 3x^2 + 2$, substitute 1 in for x. This is your slope of 5.
- Then use point slope form to find the equation of the tangent line: $y = 5x - 2$