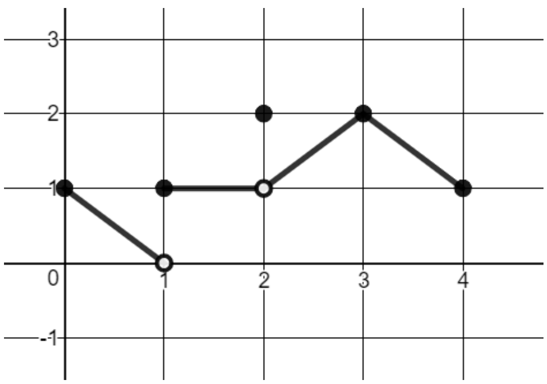
2.3 Continuity

Continuous Function:

a. A function whose outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between.
b. Any function y = f(x) whose graph can be sketched without lifting the pencil is continuous.
c. A function is continuous on an interval iff it is continuous at every point on the interval.
d. A continuous function is one that is continuous at every point of its domain.

Example:



The function is continuous on [0, 4] exempt at x = 1 and x = 2.

Points at which f is continuous:

$$\lim_{x \to 0^+} f(x) = f(0)$$

$$\lim_{x \to 4^-} f(x) = f(4)$$

$$\lim_{x \to c} f(x) = f(c)$$

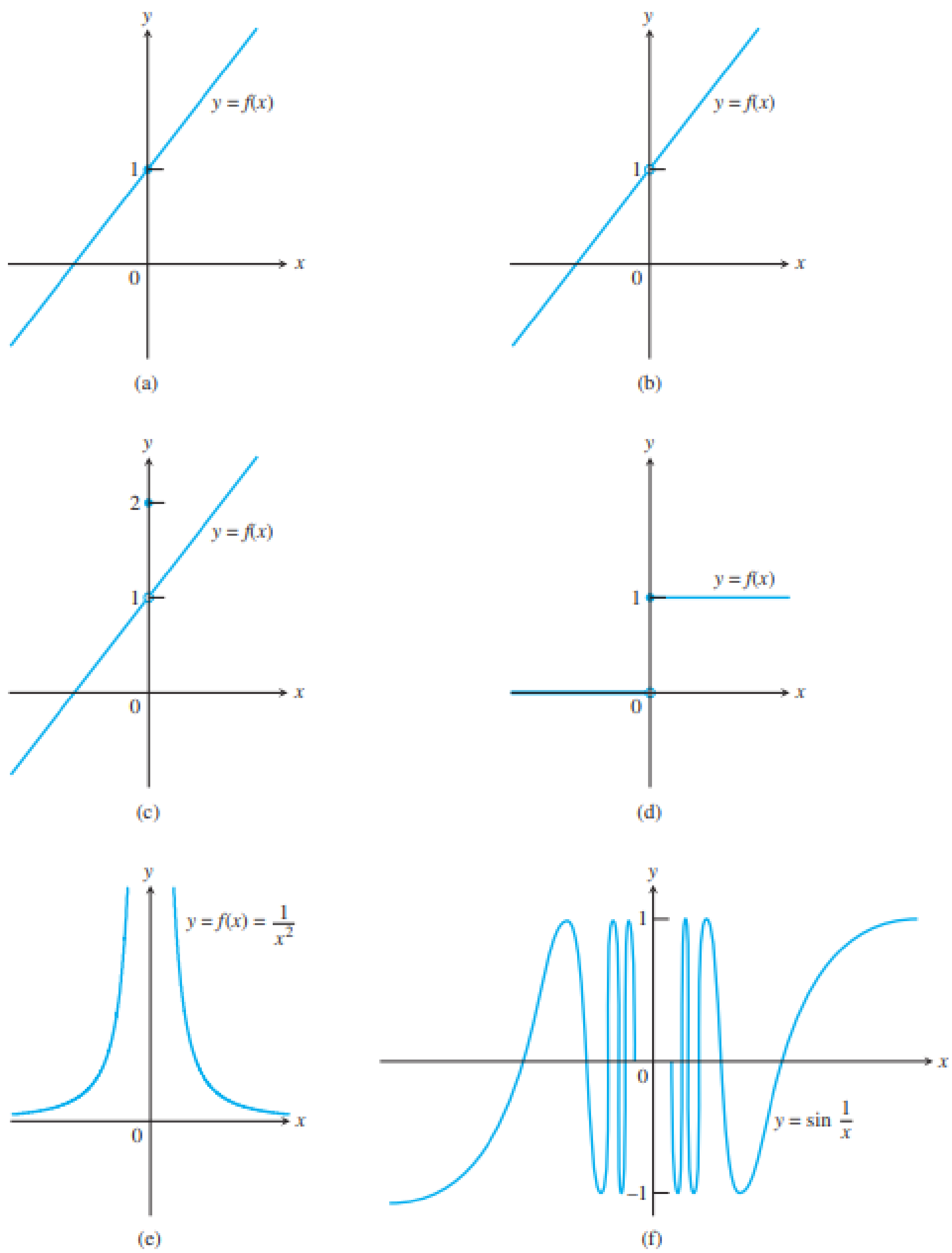
Continuity at a Point:

- Interior Point: A function $y = f(x)$ is continuous at an interior point c of its domain if $\lim_{x \to c} f(x) = f(c)$
- Endpoint: A function $y = f(x)$ is continuous at the left endpoint or is continuous at the right endpoint of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \; or \; \lim_{x \to b^-} f(x) = f(b)$$

Discontinuous: A function is discontinuous if it is not continuous at a point c.

Examples of Discontinuity:



a.) Is continuous at x = 0
b.) The discontinuity is removable.
c.) The discontinuity is removable.
d.) Has a jump discontinuity.
e.) Infinity Discontinuity
f.) Oscillating Discontinuity.

Removing a Discontinuity: Let $f(x) = \frac{x^3-7x-6}{x^2-9}$

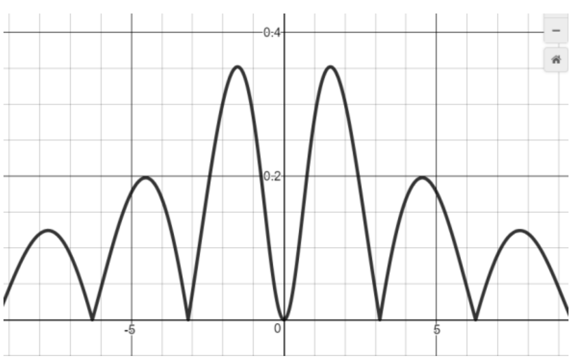
1. Factor the denominator. What is the domain of f?
2. Investigate the graph of f around x= 3 to see that f has a removable discontinuity at x =3.
3. How should f be defined at x = 3 to remove the discontinuity?
4. Show that (x-3) is a factor of the numerator of f and remove all common factors. Now compute the limit as $x \to 3$ of the reduced form for f.
5. Show that the extended function: $g(x) = \begin{cases} \frac{x^3-7x-6}{x^2-9}, & x \neq 3 \\ \frac{10}{3}, & x = 3 \end{cases}$

Properties of Continuous Functions: If the functions f and g are continuous at x = c, then the following combinations are continuous at x = c.

1. Sums: $f + g$
2. Differences: $f - g$
3. Products: $f * g$
4. Constant Multples: $k * f, for\ any\ number\ k$
5. Quotients: $\frac{f}{g}, provided\ g(c) \neq 0$

Composites: All composites of continuous functions are continuous. If f is continuous at c and g is continuous at $f(c)$, then the composite of $f \circ g$ is continuous at c.

Example: Show that $y = \left|\frac{x \sin x}{x^2+2}\right|$ is continuous.



Let $g(x) = |x|$ and $f(x) = \frac{x \sin x}{x^2+2}$
We can see that y is a composite of $g \circ f$.

We know that the absolute value of x is continuous. The function f is continuous by properties of continuous as functions. Therefore this function is continuous.

Intermediate Value Theorem for Continuous Functions:
A function $y = f(x)$ that is continuous on a closed interval [a, b] takes on every value between f(a) and f(b). In other words, if y_0 is between f(a) and f(b), then $y_0 = f(c)$ for some c in [a, b].