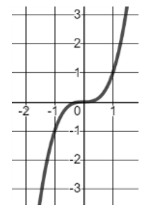
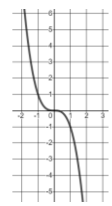
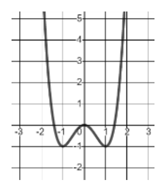
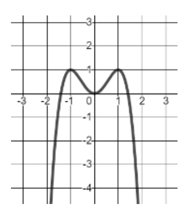
1.5 Polynomial Notes

Polynomial Function: Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_2x^2 + a_1x + a_0$$

Is called a polynomial function in x of degree n.

Graphs: We read graphs from left to right.

Leading Coefficient	What does the graph do
Odd and positive	Starts in bottom left and raises to right. 
Odd and negative	Starts in the upper left and drops right 
Even and positive	Both ends go up 
Even and negative	Both ends go down 

Real Zeros of Polynomial Functions

If f is a polynomial function and a is a real number, the following statements are equivalent.

1. $x = a$ is a zero of the function f.
2. $x = a$ is a solution of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a factor of the polynomial $f(x)$.
4. $(a, 0)$ is an x-intercept of the graph of f.

Find Zeros of a Polynomial Function
Example:
Find all real zeros of $f(x) = x^3 - x^2 - 2x$

1. Set equal to zero. $0 = x^3 - x^2 - 2x$
2. Factor an x. $0 = x(x^2 - x - 2)$
3. Factor the quadratic. $0 = x(x-2)(x+1)$
4. Solve for x. $x = 0, 2, -1$
5. Check work.

Repeated Zeros: For a polynomial function, a factor of $(x-a)^k, k > 1$, yields a repeated zero $x = a$ of multiplicity k.

1. If k is odd, the graph crosses the x-axis at x = a.
2. If k is even, the graph touches the x-axis (but does not cross the x-axis) at x = a.

Long Division of Polynomials
Divide $6x^3 - 19x^2 + 16x - 4$ *by* $x - 2$, and use the result to factor the polynomial completely.

Solution

Partial quotients

$$\begin{array}{rl} 6x^2 - 7x + 2 & \\ x-2\,\overline{)\,6x^3 - 19x^2 + 16x - 4} & \\ \underline{6x^3 - 12x^2} & \text{Multiply: } 6x^2(x-2). \\ -7x^2 + 16x & \text{Subtract.} \\ \underline{-7x^2 + 14x} & \text{Multiply: } -7x(x-2). \\ 2x - 4 & \text{Subtract.} \\ \underline{2x - 4} & \text{Multiply: } 2(x-2). \\ 0 & \text{Subtract.} \end{array}$$

$$f(x) = (x-2)(2x-1)(3x-2), \qquad x = 2, \frac{1}{2}, \frac{2}{3}$$

Sometimes, long division will produce a nonzero remainder.
Example: $x^2 + 3x + 5$ *by* $x + 1$

$$\begin{array}{rl} x + 2 & \Leftarrow \text{Quotient} \\ \text{Divisor} \Rightarrow \quad x+1\,\overline{)\,x^2 + 3x + 5} & \Leftarrow \text{Dividend} \\ \underline{x^2 + x} & \\ 2x + 5 & \\ \underline{2x + 2} & \\ 3 & \Leftarrow \text{Remainder} \end{array}$$

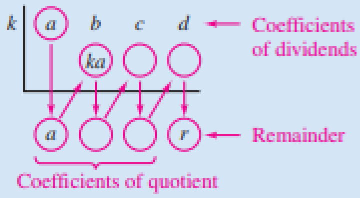
In fractional form, you can write this result as follows.

$$\overbrace{\frac{x^2 + 3x + 5}{\underbrace{x+1}_{\text{Divisor}}}}^{\text{Dividend}} = \overbrace{x + 2}^{\text{Quotient}} + \frac{\overset{\text{Remainder}}{3}}{\underbrace{x+1}_{\text{Divisor}}}$$

Synthetic Division:

Synthetic Division (of a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.

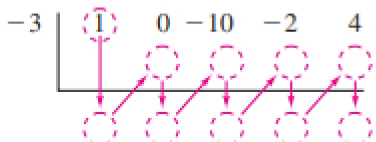


Vertical pattern: Add terms.
Diagonal pattern: Multiply by k.

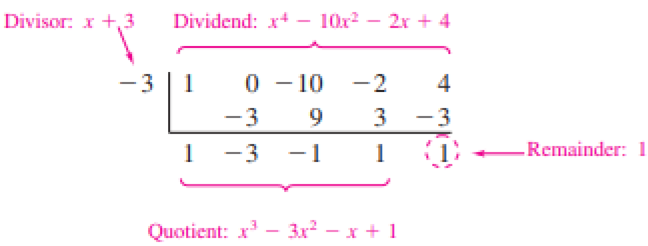
Use synthetic division to divide $x^4 - 10x^2 - 2x + 4\ by\ x + 3$

Solution

You should set up the array as follows. Note that a zero is included for each missing term in the dividend.



Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3.



So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

The Rational Zero Test
If the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_2 x^2 + a_1 x + a_0$$

Has integer coefficients, every rational zero of f has the form

$$Rational\ zero = \frac{p}{q}$$

Where p and q have no common factors other than 1, p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

Example: Find the rational zeros of $f(x) = 2x^3 + 3x^2 - 8x + 3$

$$p = 2\ and\ q = 3$$

$$Possible\ Rational\ Zeros = \frac{factors\ of\ 3}{factors\ of\ 2} = \ \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$f(x) = (x - 1)(2x^2 + 5x - 3) = (x - 1)(2x - 1)(x + 3)$$